

# Fast Fourier and Wavelet Transforms for Wavefront Reconstruction in Adaptive Optics

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# Fast Fourier and Wavelet Transforms for Wavefront Reconstruction in Adaptive Optics

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## ABSTRACT

Wavefront reconstruction techniques using the least-squares estimators are computationally quite expensive. We compare wavelet and Fourier transforms techniques in addressing the computation issues of wavefront reconstruction in adaptive optics. It is shown that because the Fourier approach is not simply a numerical approximation technique unlike the wavelet method, the Fourier approach might have advantages in terms of numerical accuracy. However, strictly from a numerical computations viewpoint, the wavelet approximation method might have advantage in terms of speed. To optimize the wavelet method, a statistical study might be necessary to use the best basis functions or "approximation tree."

**Key words:** Novel algorithms and architectures, adaptive optics, wavefront reconstruction, fast algorithms, real-time imaging, wavefront control, phase reconstruction, FFT methods, wavelets, MEMS.

## 1. INTRODUCTION

Wavefront reconstruction techniques for real-time imaging applications using adaptive optics are computationally intensive. As we approach implementation of adaptive optics systems with thousands of actuators employing micro-electromechanical systems (MEMS), the use of efficient high-performance algorithms for wavefront reconstruction will be of significant importance. In this study we have evaluated various fast wavelet and Fourier methods in order to determine both the speed and robustness of wavefront reconstruction algorithms for real-time imaging in large scale adaptive optics systems.

We developed fast signal processing algorithms for wavefront reconstruction. In particular, we have studied both fast discrete wavelet (DWT) and fast Fourier methods (FFT) to determine the performance of these algorithms in wavefront phase reconstruction. The wavelet method is of particular interest because the order of the computation for the 1-D DWT is only  $O(N)$  as opposed to  $O(N \log N)$  for the 1-D FFT. The usefulness of the wavelets in terms of reduced number of computations is enhanced by the fact that the transform coefficients can be truncated significantly; i.e. transformed into sparse expansions. This energy compaction property of the 2-D wavelet transforms can be exploited fully in the wavefront reconstruction algorithm in adaptive optics. Since there are many "good wavelets" or basis functions, in this study we compare the performance for wavelets for fast and robust wavefront reconstruction.

In section 2 we first derive the Fourier approach and show that this approach is really an analytical alternative to the true solution. On the other hand, in section 3 we show that the wavelet method is

strictly a numerical approximation to the least square technique. We illustrate our results with the wavelet scheme, and summarize our conclusions in section 4.

## 2. WAVEFRONT RECONSTRUCTION USING THE FFT

If we sample a wavefront on a regular 2D grid we have an array of phase values  $\varphi_{mn}$  where  $0 \leq m \leq M-1$  and  $0 \leq n \leq N-1$ . The phase differences  $\Delta_{m,n}^x$  and  $\Delta_{m,n}^y$  are then given by

$$\Delta_{m,n}^x = \varphi_{m+1,n} - \varphi_{m,n}$$

$$\Delta_{m,n}^y = \varphi_{m,n+1} - \varphi_{m,n}$$

We can write the phase differences as a matrix equation

$$\bar{\Delta} = A \bar{\varphi}$$

where  $\bar{\Delta}$  is a column vector containing the x and y phase differences,  $\bar{\varphi}$  is a column vector containing the phase values, and  $A$  is a finite difference matrix. The least-squares solution for  $\bar{\varphi}$ , given a slope measurement vector  $\bar{\Delta}$ , is given by solving the normal equations

$$A^T A \bar{\varphi} = A^T \bar{\Delta}$$

In most adaptive optics systems the normal equations are solved directly using singular-value decomposition to give the phase estimate

$$\hat{\varphi} = (A^T A)^{-1} A^T \bar{\Delta}$$

This solution requires  $(MN)^2$  operations to implement the real-time phase reconstructor, assuming that the matrix  $(A^T A)^{-1} A^T$  is pre-computed.

For large-scale systems with thousands of phase sample points we need a more computationally efficient approach. The normal equations can be written in the form [3]:

$$(\varphi_{m+1,n} - 2\varphi_{m,n} + \varphi_{m-1,n}) + (\varphi_{m,n+1} - 2\varphi_{m,n} + \varphi_{m,n-1}) = \rho_{m,n}$$

where

$$\rho_{m,n} = (\Delta_{m,n}^x - \Delta_{m-1,n}^x) + (\Delta_{m,n}^y - \Delta_{m,n-1}^y)$$

Note that this is a discretization of Poisson's equation  $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = \rho$ . The phase  $\varphi_{m,n}$  can be represented by its discrete Fourier coefficients as

$$\varphi_{m,n} = \frac{1}{N} \sum_{p,q} \tilde{\varphi}_{p,q} e^{i \frac{2\pi}{N}(pm+qn)}$$

where we have assumed that  $M = N$ , i.e. the phase sample grid is square. We also have

$$\varphi_{m+1,n} = \frac{1}{N} \sum_{p,q} \tilde{\varphi}_{p,q} e^{i \frac{2\pi p}{N}} e^{i \frac{2\pi}{N}(pm+qn)}$$

Now we can write the normal equations in terms of the Fourier coefficients as

$$\begin{aligned} & \sum_{p,q} \tilde{\varphi}_{p,q} e^{i \frac{2\pi}{N}(pm+qn)} \left( e^{i \frac{2\pi p}{N}} - 2 + e^{-i \frac{2\pi p}{N}} \right) + \sum_{p,q} \tilde{\varphi}_{p,q} e^{i \frac{2\pi}{N}(pm+qn)} \left( e^{i \frac{2\pi q}{N}} - 2 + e^{-i \frac{2\pi q}{N}} \right) \\ &= \sum_{p,q} \tilde{\rho}_{p,q} e^{i \frac{2\pi}{N}(pm+qn)} \end{aligned}$$

where  $\tilde{\rho}_{p,q}$  are the Fourier coefficients of  $\rho_{m,n}$ . For spatial frequency  $p,q$  we can solve directly for  $\varphi_{p,q}$  as

$$\tilde{\varphi}_{p,q} = \frac{\tilde{\rho}_{p,q}}{2 \left( \cos \frac{2\pi p}{N} + \cos \frac{2\pi q}{N} - 2 \right)}$$

Now we need to write  $\tilde{\rho}_{p,q}$  in terms of the measurements  $\Delta_{m,n}$ .

$$\tilde{\rho}_{p,q} = \Im \{ \Delta_{m,n}^x \} - \Im \{ \Delta_{m-1,n}^x \} + \Im \{ \Delta_{m,n}^y \} - \Im \{ \Delta_{m,n-1}^y \}$$

where  $\Im \{ f \}$  is the Fourier transform of  $f$ . Since  $\Im \{ \Delta_{m-1,n}^x \} = e^{-i \frac{2\pi p}{N}} \tilde{\Delta}_{p,q}^x$  we have

$$\tilde{\rho}_{p,q} = \left( 1 - e^{-i \frac{2\pi p}{N}} \right) \tilde{\Delta}_{p,q}^x + \left( 1 - e^{-i \frac{2\pi q}{N}} \right) \tilde{\Delta}_{p,q}^y$$

Then we can write the Fourier coefficients of the phase estimate as the sum of two spatial filter operations

$$\tilde{\varphi}_{p,q} = H_{p,q}^x \tilde{\Delta}_{p,q}^x + H_{p,q}^y \tilde{\Delta}_{p,q}^y$$

where the filters are given by

$$H_{p,q}^x = \frac{e^{-i\frac{2\pi p}{N}} - 1}{4 \left( \sin^2 \frac{\pi p}{N} + \sin^2 \frac{\pi q}{N} \right)}$$

$$H_{p,q}^y = \frac{e^{-i\frac{2\pi q}{N}} - 1}{4 \left( \sin^2 \frac{\pi p}{N} + \sin^2 \frac{\pi q}{N} \right)}$$

The algorithm for the Fourier wavefront reconstructor is given by

1. Compute the Fourier transforms of the phase differences from the wavefront sensor,  $\tilde{\Delta}_{p,q}^x, \tilde{\Delta}_{p,q}^y$ ,
2. Apply the spatial filters  $H_{p,q}^x, H_{p,q}^y$  to get  $\tilde{\varphi}_{p,q}$
3. Inverse Fourier transform to get  $\varphi_{m,n}$

If  $N$  is a power of two the spatial filter operations can be implemented with FFT's. The computational requirements then scale as  $O(N^2 \log_2 N)$  rather than as  $O(N^4)$  in the direct vector-matrix multiplication approach.

### 3. WAVEFRONT RECONSTRUCTION USING WAVELETS

Although wavelets are well known for their signal and image compression properties [2,5,6], a less well-known use of the wavelet transform is in obtaining a fast approximate numerical solution [3] for a system of linear equations. As discussed in the previous section, this implies wavelets can be used in the adaptive optics wavefront reconstruction problem. The matrix operator, in solving a system of linear equations, can be thought of as a two-dimensional image. Taking advantage of the energy compaction property of the two-dimensional wavelet transform, we can expect a large fraction of the wavelet coefficients to be small and negligible. Hence, the linear system to be solved, in the transform domain, is a sparse system (mostly zero coefficients). To be more specific, consider solving a set of linear equations of the form:

$$Ax=b$$

The two-dimensional wavelet transform of the system matrix,  $A$ , and the vector  $b$  are denoted by:

$$\tilde{A}=WAW^T$$

$$\tilde{b}=Wb$$

where  $W$  represents the kernel of the wavelet transform operator. Now we can solve for

$$\tilde{A}\tilde{x}=\tilde{b}$$

where  $\tilde{x}$  represents the wavelet transform of the solution vector. To obtain the solution  $x$ , we take a final inverse of the orthogonal wavelet transform:

$$x = W^T \tilde{x}$$

The efficiency of the wavelet transform in solving a set of linear equations will ultimately depend both on the sparseness of the matrix  $\tilde{A}$  and on the particular wavelet transform approximation algorithm. For example, consider the 2-D wavelet transform depicted in Figure 1.

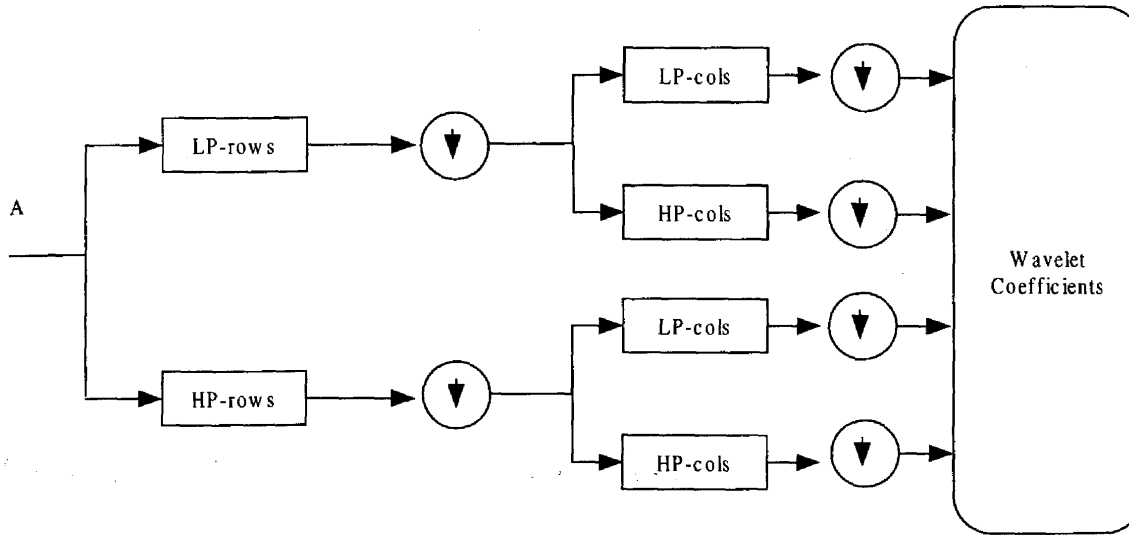


Figure 1: A block diagram of the forward wavelet transform for two-dimensional waveforms or images. Note that for the inverse transform, simply replace downsampling (down-arrows) by upsampling (up-arrows) and right-arrows by left arrows.

The accuracy of the numerical approximation and the number of computation will depend on the number of decomposition steps and will further depend on the filterbanks “tree” or path employed for the transformation [5]. There are also a number of different approximation trees that can be chosen from; i.e. the exact path of lowpass and highpass filters. We studied lowpass, highpass, and both lowpass and highpass approximations for adaptive optics wavefront reconstruction. In our implementation scheme, we found that the lowpass wavelet coefficients were generally more useful. The results in this paper are based on the double lowpass wavelet approximation coefficients. The inverse wavelet transform is very similar to the forward transform shown in Figure 1, except that the down-sampling by two (down-arrows) are replaced by up-sampling (placing zeros in between data samples) and the right-arrows are to be replaced by left arrows. Finally, the set of filter coefficients used to compute the wavelet transforms and the inverse wavelet transforms is shown in Figure 2.

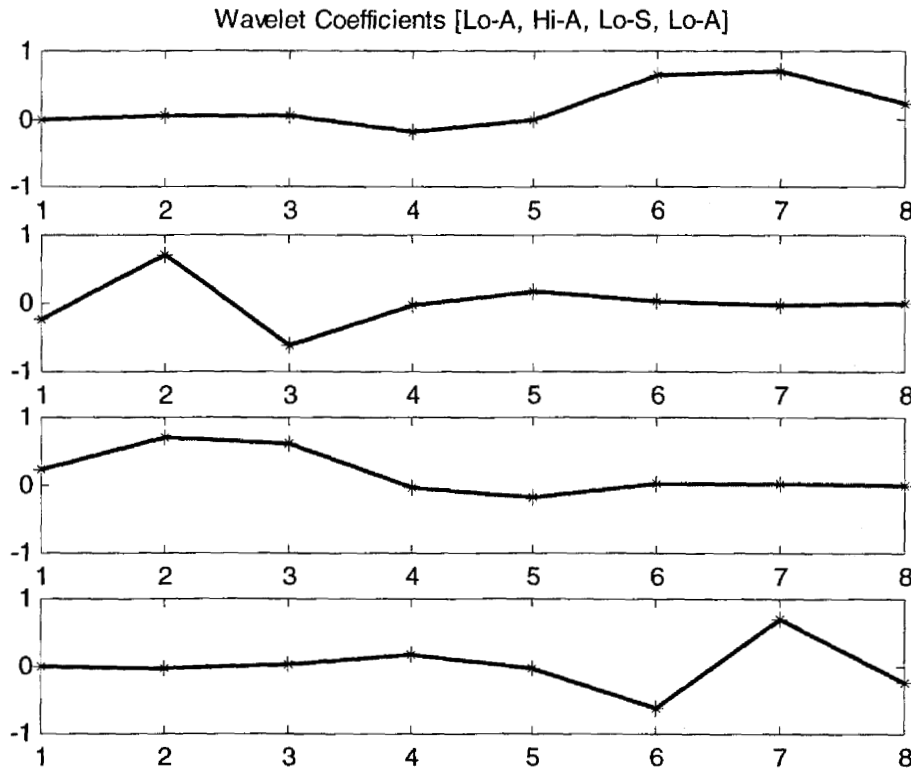


Figure 2: Plots of the typical wavelet transform filter coefficients: lowpass analysis, highpass analysis, lowpass reconstruction, and highpass reconstruction filters.

In summary, the algorithm can be described as follows:

Step 1: Compute the wavelet matrix approximation for a single stage wavelet transform. This step is precomputed just once, and depends on the system.

Step 2: This step is repeated multiple times:

- 2.1 *Compute the wavelet transform of the input vector;*
- 2.2 *Compute multiplication in the wavelet domain;*
- 2.3 *Reconstruct the vector approximation by taking the inverse wavelet transform.*

Note that because the wavelet transform basis has only a small number of terms (localized) and because of the downsampling operations, it can be shown that the computation of the wavelet transform is  $O(M)$  in 1-D and  $O(M^2)$  in 2-D, where  $M$  is the dimension of the input. The computational requirements for the wavelet wavefront reconstruction approach then scales as  $O(N^2)$  rather than as  $O(N^4)$  in the direct vector-matrix multiply approach, and  $O(N^2 \log_2 N)$  using the FFT method. A note of caution: although the wavelet transform is numerically most attractive, it is important to point out that this approach is an *approximation* and its usefulness will ultimately depend on how well it does in terms of numerical accuracy. In the next section we present results from applying the wavelet reconstruction method of real wavefront data.

## 4. NUMERICAL EXAMPLES USING WAVELETS

In section 4.1 we first illustrate the wavelet approximation scheme using a simple tractable numerical example. Performance of the approximation on real data is illustrated in sections 4.2 and 4.3. Finally, comments on areas of care required for using this method are included.

### 4.1 A SIMPLE NUMERICAL ILLUSTRATION

For the sake of clarity we first consider simple numerical example on matrix-vector multiplication using the wavelet transform scheme. Consider a (8x8) matrix A, and a (8x1) column vector x shown below:

$$A = \begin{bmatrix} 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \\ 9 & 55 & 54 & 12 & 13 & 51 & 50 & 16 \\ 17 & 47 & 46 & 20 & 21 & 43 & 42 & 24 \\ 40 & 26 & 27 & 37 & 36 & 30 & 31 & 33 \\ 32 & 34 & 35 & 29 & 28 & 38 & 39 & 25 \\ 41 & 23 & 22 & 44 & 45 & 19 & 18 & 48 \\ 49 & 15 & 14 & 52 & 53 & 11 & 10 & 56 \\ 8 & 58 & 59 & 5 & 4 & 62 & 63 & 1 \end{bmatrix}, x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$

The numerical solutions to the direct matrix-vector multiplication and the wavelet approximation are respectively given by

$$Ax = [1162, 1178, 1178, 1162, 1162, 1178, 1178, 1162]^T$$

$$w^{-1}[\tilde{A} \tilde{x}] = [1170, 1170, 1170, 1170, 1170, 1170, 1170, 1170]^T$$

Note that in the wavelet approximation method, we used the 2<sup>nd</sup> order Daubechies wavelet coefficients with two consecutive lowpass (or smooth approximation) filtering and downsampling of the rows and the columns. The solution therefore represents a smooth approximation of the exact solution. In general, this is an important point about the wavelet approximation method on real data: the approximation quality depends on the exact decomposition tree (i.e. the path of lowpass and highpass approximations) and on determining how well a particular tree is able to represent the matrix A and therefore also the solution vector Ax.

### 4.2 EXAMPLES WITH REAL DATA

An example of the AO system matrix is shown as an image in Figure 3. To illustrate the approximation with a discernable plot, we show part of the reconstruction result in Figure 4, and

compare this wavelet approximation solution with the conventional reconstruction technique. In this particular example, we used a linear ramp signal for  $x$ , the slope measurement vector.

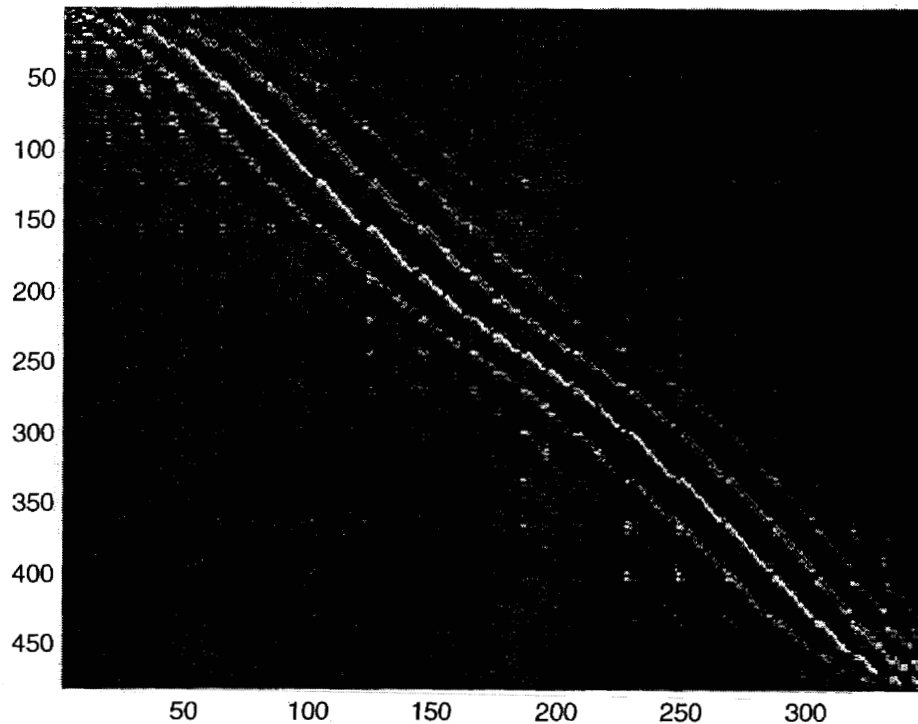


Figure 3: The AO System Matrix (A) represented as an image

Although the wavelet transform technique is computationally highly efficient, through the course of this study we discovered that the approximation quality does depend on the form of system matrix,  $A$ , and on the multiplying vector,  $x$ . Furthermore, the exact lowpass-highpass combinations used in the approximation algorithm is also quite important. Our conclusion is that to use the wavelet transform technique successfully, we would need to first carefully characterize the statistical properties of the distorted wavefronts and then select the best "basis" (the lowpass/highpass tree). A significant amount of work has been performed in wavelet packets [6] that could be used for optimal performance. Finally, we plan to evaluate the performance of this in a full adaptive optics system simulation mode.

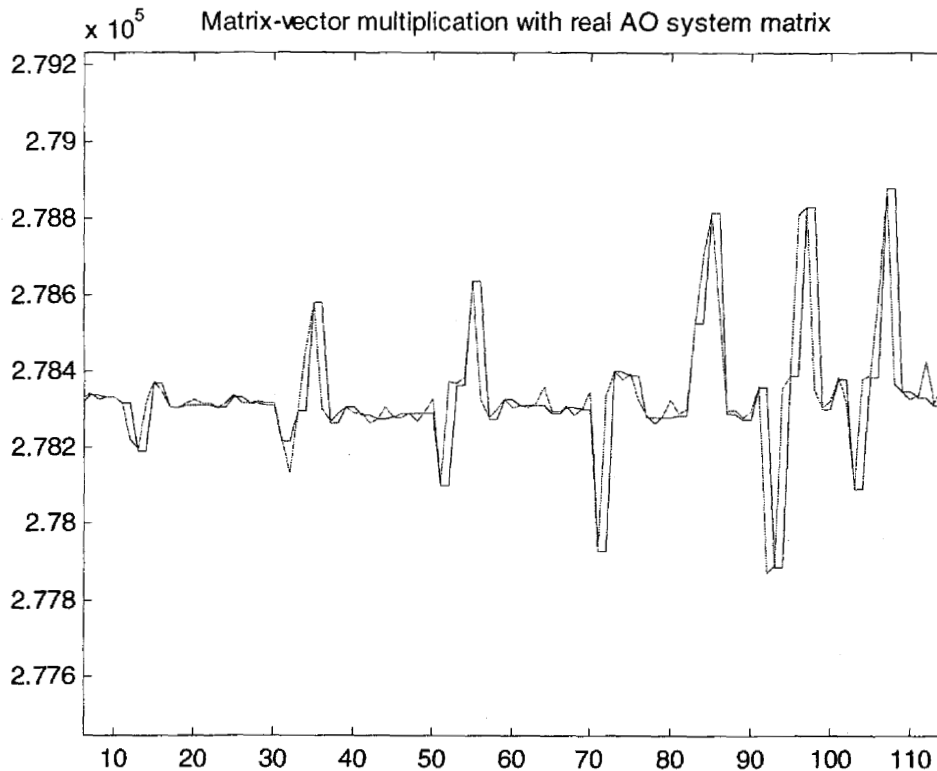


Figure 4: The matrix-vector multiplication result  $y = Ax$ , where  $A$  is shown in Figure 1 and  $x$  is a linear ramp vector [1 2 ... 349]. Only a section of 112 points of the solution vector and the corresponding wavelet approximate smooth solution is plotted over. The smoother graph represents the wavelet approximation solution.

## 5. SUMMARY AND CONCLUSIONS

In this paper we discussed how the FFT and the wavelet transform methods could be used to speed up computation in adaptive optics wavefront reconstruction. We were particularly interested in comparing wavelet method with the FFT technique. Although, in terms of the number of operations, the wavelet transform seems to be more attractive as discussed in sections 2 and 3, we find that the wavelet approximation might or might not be accurate enough. Only a full scale simulation study can determine the performance in terms numerical accuracy; we plan to do this in the near future. The accuracy of the approximation depends on the particular vector and matrix properties, and on the choice of the wavelet tree used to decompose the tree. A statistical study will be needed to select the optimal wavelet decomposition. On the other hand, as discussed in section 2, the Fourier approach is much less of a numerical approximation, and more of an analytical solution until the application of the FFT. Hence, the FFT approach might be more suitable to minimize loss in the numerical accuracy.

We observed that the type of wavelet basis function used does not seem to make a significant difference in the numerical results. A low order basis function, like the Daubechies 2<sup>nd</sup> order coefficients, seems to be adequate with real data.

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